Lee Overlay Partners

CURRENCY OVERLAY MANAGEMENT

Research Report

Options or forwards? It's all the same in the long run

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Abstract

In this article we simplify the decision of using options or forwards in order to hedge an unwanted exposure for investors with a long term horizon by showing that a sequence of option payoffs converge in time to a payoff that can be realised using forwards. The forwards are not used in a dynamic sense, such as delta hedging, but in a static buy-and-hold sense. The key result is that although a single option provides an asymmetric payoff, a sequence of them converges to a symmetric payoff and long run protection is not therefore provided.

1. Introduction

Suppose we own a security that creates unwanted risk in our portfolio. Currency exposure in an international portfolio is an example. In order to reduce, or eliminate, that risk we can hedge it away with derivative instruments. A typical hedging strategy is to use forward contracts to sell a proportion of the security in the future at a price specified today. Another popular strategy is to buy a put option to insure against price depreciation but participate in any appreciation. For details on strategic currency investing, the reader is referred to Gitlin (1993) and Thomas (1990) and references therein.

Investors who are required to manage this unwanted risk are faced with the decision of which strategy to employ. Hedge with forwards or options. We focus on an investor who has a long run horizon, an institutional pension plan sponsor for example. Such an investor must decide whether to use forwards to hedge a certain proportion of the security and maintain that hedge throughout the life of the portfolio by rolling the forward contracts, or pay the premia for a sequence of protective put options.

In this article we simplify the option versus forward decision by showing that in the long run the put protection strategy yields an identical return distribution to the forward hedging strategy. We will show that the proportion of forward hedging required to match the put protection depends on the strike of the option and is approximately equal to the initial delta of the put protected portfolio. This result has a nice intuitive interpretation in that a deep out-of-the-money put strategy equates to an unhedged portfolio; a deep in-the-money put strategy equates to a fully hedged portfolio, and an at-the-money put strategy equates to a half hedged portfolio. More generally we have a useful duality in that if we know what proportion of hedging is desired, we can derive the strike price of the option needed to produce it. Conversely, if we know the strike price of the put option strategy, we can derive the appropriate hedging proportion to produce an equivalent return distribution.

We know that by dynamically changing the proportion of the forward hedging we can mimic the long-run put protected strategy returns (in fact this is a well-used technique in currency management). It is perhaps more surprising to find that a static hedge proportion of forward hedging can also mimic the long-run return distribution of put protection.

In Section 2 we provide an application of our result. In Section 3 we give an empirical example. In Section 4 we support our theory with a simulation. In Section 5 we give the mathematical proof of our theory. In Section 6 we discuss a possible arbitrage between options and forwards. In Section 7 we provide a conclusion of our findings.

2. An application of our result: The strategic hedging policy

The construction of a portfolio to maximise long term objectives is known as strategic asset allocation. The standard approach to making the strategic decision is to use mean/variance optimisation (Markowitz 1952), where required inputs are the long term expected returns, volatilities, and correlations of the assets. Given these inputs, the mean/variance optimisation computes the vector of portfolio weights that maximise the portfolio expected return for a specified level of portfolio risk. The maximisation can incorporate any investment constraints.

Mean/variance optimisation is very simple to implement, but statistical decision theory tells us that the formal solution to strategic asset allocation is to maximise the investor's expected utility of wealth (von Neuman and Morgenstern 1947). It has been shown that the two approaches are identical when the asset returns are normally distributed (Tobin 1958 and Feldstein 1969).

However, option returns are very asymmetric and therefore not normally distributed. For this reason options have not been incorporated into a typical strategic study. But in this article we show that in the long run they are indeed normal and can therefore be included in the mean/variance optimisation.

If there are international assets in the portfolio, then the strategic allocation will involve a currency decision. It is not uncommon to address the strategic currency decision in a separate study. Long run currency returns appear to be normally distributed (Lee Overlay Partners 2000) so the using mean/variance optimisation is valid for strategic hedging analyses (even if the short-term currency returns are not normally distributed).

Strategic studies reflect a neutral expected return outlook, and in the case of currencies the neutral outlook is that the currency will follow uncovered interest rate parity (see Krugman and Obstfeld 1991 for details). With such an outlook the option protected strategy is identical to a partially hedged portfolio (see Section 5 for a proof).

The implication is that the strategic study becomes a two-stage process.

Stage 1:

Do a standard mean/variance study to derive the optimal amount of currency exposure to be left in the portfolio, denoted by α.

Stage 2:

Decide whether the hedging is to be done with options and forwards, or a mixture of them both. If options are chosen then the appropriate option delta can be deduced via the equation

$$\alpha \approx \tfrac{1}{2}(\Delta+\Delta^{1/2})$$

where Δ is the delta of the protected portfolio.

We have plotted the relationship between hedge ratio and delta in Chart 1 where the currency volatility is set at 10%. We can clearly observe the intuitive results that deep out-of-the money put protection is equivalent to an unhedged portfolio, deep in-the-money put protection is equivalent to a fully hedged portfolio, and at-the-money put protection is similar to a half hedged portfolio.

From Chart1 we can see that given a delta we can derive the hedge ratio that yields an identical long run return distribution, and conversely given a hedge ratio we can derive the corresponding delta. Also in Chart 1 we can see the accuracy of our approximation $\alpha \approx \frac{1}{2}(\Delta+\Delta^{\frac{1}{2}})$ as both the approximated curve and the true curve are plotted and seen to be almost indistinguishable.

3. An example using empirical data

We examine a portfolio of currencies (30% JPY, 40% EUR, 30% GBP) and compute the average return and the risk of that portfolio for a variety of hedge ratios from fully hedged to unhedged. These portfolios appear as a sequence of points on the return/risk plot in Chart 2.

We also use historical implied volatility data to derive the returns of an option protected portfolio. We use options with a variety of deltas from 0 to 1. These portfolios also appear as a sequence of points on the return/risk plot on Chart 2.

The important conclusion is that there is no significant difference between the long run return profiles of the forwards or the options. Therefore the long run investor should be indifferent to the use of either instrument (although the higher transactions costs of the option strategies may make the forwards a more attractive alternative).

4. Illustration by simulation

Given the complex nature of the mathematics underlying our analysis, the easiest way to illustrate the convergence property of the options and forwards is by simulation. As an example we simulate a currency price process with a zero risk free rate differential, 10% implied volatility, and instantaneous mean equal to the risk free differential. We use at-the-money options struck annually.

After one year, the downside protection from the put option produces an asymmetric distribution and is therefore quite distinct from a static proportional forward hedging strategy. We simulated 10,000 one-year returns from the option strategy and plotted them in the first of three histograms in Chart 3. The asymmetry of the option strategy can be seen, particularly when measured against the distribution of returns from the forwards strategy with a 40% hedge ratio, represented in Chart 3 by the superimposed solid line.

After several years, the put protection downside limit becomes more negative as the single year downside limits are compounded. For example after five years the distribution is much more similar to a static hedging strategy as the histogram of 10,000 simulated five-year returns in Chart 3 illustrates. By the time ten years have passed there is little difference between the forward hedging and the put protection. The third histogram in Chart 3 illustrates how close the strategies have become in a simulation of 10,000 ten-year returns.

5. A mathematical approach

5.1 Introduction

The simulation approach provides a great deal of insight into the nature of our discussion. Even in the isolated example cited above, the convergence of the option based returns to the forwards based returns is clearly exhibited. However, in order to firmly claim that the two approaches are equivalent, it is incumbent upon us to provide a mathematical proof of this proposition.

In our mathematical approach we will prove that the long-run return distribution of the option strategy is equal to the long-run return distribution of the forwards strategy for a specified hedging proportion. To do this we will show that the distributions are identical in class then demonstrate that the mean and variances of the two strategies are identical.

5.2 Defining the portfolios

We analyse three portfolios. The first portfolio is simply the risky asset. The second represents the forwards based strategy. The third represents the option protected strategy.

Portfolio 1:

Portfolio 1 holds the risky asset. We denote the price of the risky asset, and hence first portfolio, at time t by Z_t.

Portfolio 2:

Portfolio 2 holds the risky asset and forward contracts to hedge a proportion of the asset. We denote the price of the second portfolio at time t by X_t, and note that

$$X_t / X_{t-1} = (1 - \alpha) f_{t,t-1} / Z_{t-1} + \alpha\, Z_t / Z_{t-1}$$

where $f_{t,t-1}$ denotes the forward rate at time t maturing at time $t+1$, and $(1 - \alpha)$ is the proportion of the portfolio that has been hedged.

Portfolio 3:

Portfolio 3 holds the risky asset and a put option. We denote the price of the third portfolio at time t by Y_t, and note that

$$Y_t / Y_{t-1} = \max(Z_t / Z_{t-1}, K_{t-1} / Z_{t-1}) + p_{t-1} e^r / Z_{t-1}$$

where p_{t-1} and K_{t-1} denote the option price and the strike price respectively, both at time $t-1$, and r denotes the risk free rate. The variable K_{t-1} is derived from the initial delta of the put option. The initial delta will be the same for each of the options.

5.3 The portfolio returns

We are interested in the long run return distributions of these portfolios, and hence focus on the distributions of

$$\ln(Z_T / Z_0), \ln(X_T / X_0), \text{ and } \ln(Y_T / Y_0)$$

as T tends to infinity.

We can consider these long run returns as the summations of many short run returns. This is expressed mathematically by the relationship

$$\ln(S_T / S_0) = \Sigma_{t=1,T} \ln(S_t / S_{t-1})$$

that holds for any price series S_t.

The benefit of this relationship is that we already have expressions for X_t / X_{t-1}, and Y_t / Y_{t-1} that we derived in Section 5.2.

5.4 Asset return characteristics

We begin with the security price process, denoted by Z_t, which we assume follows geometric Brownian motion hence

$$dZ = \mu Z dt + \sigma Z dW$$

where dW is Brownian motion. This is the standard price process assumption for option pricing (see for example Hull 1993). Applying Ito's Lemma to this process we have that

$$\ln(Z_t / Z_{t-1}) \sim N(\mu - \tfrac{1}{2}\sigma^2, \sigma^2).$$

There are several characteristics of $\ln(Z_t / Z_{t-1})$ that will be useful subsequently. We observe that both the expected return and the variance are finite. We assume that μ and σ are not a function of time so that for all t the log returns, $\ln(Z_t / Z_{t-1})$, are identically distributed. The final characteristic of relevance to us is that the geometric Brownian motion process yields log returns that are mutually independent.

Incidentally, it is easy to show that Z_t / Z_{t-1} are log normally distributed with mean e^{μ} and variance $e^{2\mu}(\exp\{\sigma^2\}-1)$ (see Hull 1993). The variables Z_t / Z_{t-1} are therefore also independent and identically distributed.

5.5 The distribution of the long run portfolio returns

The long run return of all three portfolios is normally distributed, and the annualised expected return and risk of each portfolio are equal to the single-period expected return and risk. See Appendix A.1 for the proof of this statement.

This is a crucial result. It suggests that all three portfolios converge to a symmetric return distribution in the long run, even the option based portfolio. Therefore rolling short-term options does not provide downside protection in the long run, unless a single option of long run maturity is used.

5.6 An approximation to the log returns

At this point we should analyse the mean and variances of $\ln(Z_t/Z_{t-1})$, $\ln(X_t/X_{t-1})$, and $\ln(Y_t/Y_{t-1})$ in order to complete our study. However, these distributions are difficult to work with, and it is in fact a good deal easier to implement the approximation

$\ln(1+z) \approx z$. In our analysis z will take relatively small values, which is when this approximation is most accurate. We acknowledge the fact that our results are now subject to this approximation, but the analytical benefits are large.

5.7 The portfolio expected returns and volatilities

In this section we will derive the expected returns and volatilities of the three portfolios. To be more precise we will derive the expected excess return, which is defined as the expected return in excess of the risk free rate. As a notational point, we denote the expected excess return of price process S as μ_S and define it as

$$\mu_S = E[S_t/S_{t-1} - 1] - (e^r - 1)$$

where E[ ] denotes the ordinary expectation. We also let σ_S denote the volatility of S and it is therefore defined as

$$\sigma_S = V[S_t/S_{t-1} - 1]^{½}.$$

<u>Portfolio 1:</u>

As shown in Section 5.4, directly from log normal distribution theory we have that

$$E[Z_t/Z_{t-1}] = e^{\mu} \text{ and } V[Z_t/Z_{t-1}] = e^{2\mu}(\exp\{\sigma^2\}-1).$$

We thus have immediately that $\mu_Z = e^{\mu} - e^r$ and $\sigma_Z = e^{\mu}(\exp\{\sigma^2\}-1)^{½}$.

<u>Portfolio 2:</u>

Recalling the equation relating X to Z given in the proof of Section 5.5, we can readily derive the mean and variance of X_t/X_{t-1} from the equation

$$X_t / X_{t-1} = (1 - \alpha)\, e^r + \alpha\, Z_t / Z_{t-1}$$

where r is the risk free rate.

From this we have that

$$E[X_t / X_{t-1}] = (1 - \alpha)\, e^{r} + \alpha\, E[Z_t / Z_{t-1}] \text{ and } V[X_t / X_{t-1}] = \alpha^2\, V[Z_t / Z_{t-1}].$$

With some algebraic rearrangement we can show that

$$\mu_X = \alpha\, \mu_Z \text{ and } \sigma_X = \alpha\, \sigma_Z.$$

In conclusion we observe the ratios of the expected excess returns and volatilities of portfolios one and two, and note that

$$\mu_X / \mu_Z = \sigma_X / \sigma_Z = \alpha.$$

Portfolio 3:

We begin by relating Y to Z using the equation given in the proof of Section 5.5, and thus have that

$$Y_t / Y_{t-1} = \max(Z_t / Z_{t-1}, K_{t-1} / Z_{t-1}) + p_{t-1}\, e^{r} / Z_{t-1}.$$

where p_{t-1} denotes the price of the put option at time $t-1$.

Given that the option is priced as the present value of the expected future value of the option using a growth rate equal to the risk free rate, we can derive $E[Y_t / Y_{t-1} -1]$ using exactly the same mathematics.

In Appendix A.2(i) we show that

$$E[Y_t / Y_{t-1} -1] = e^{\mu}\Phi(\delta_1) + (K_{t-1} / Z_{t-1})\Phi(-\delta_2) - (e^{r}\Phi(d_1) + (K_{t-1} / Z_{t-1})\Phi(-d_2) - e^{r}) - 1$$

where

$$d_1 = (\ln(Z_{t-1} / K_{t-1}) + r + \tfrac{1}{2}\sigma^2)/\sigma \text{ and } d_2 = (\ln(Z_{t-1} / K_{t-1}) + r - \tfrac{1}{2}\sigma^2)/\sigma,$$

and

$$\delta_1 = (\ln(Z_{t-1} / K_{t-1}) + \mu + \tfrac{1}{2}\sigma^2)/\sigma \text{ and } \delta_2 = (\ln(Z_{t-1} / K_{t-1}) + \mu - \tfrac{1}{2}\sigma^2)/\sigma.$$

The variance of portfolio 3 can be derived in a similar fashion. In Appendix A.2(ii) we show that

$$V[Y_t/Y_{t-1} - 1] = e^{2\mu}\exp\{\sigma^2\}\Phi(\delta_0) + (K_{t-1}/Z_{t-1})^2\Phi(-\delta_2) - (e^{\mu}\Phi(\delta_1) + (K_{t-1}/Z_{t-1})\Phi(-\delta_2))^2.$$

where $\delta_0 = (\ln(Z_{t-1} / K_{t-1}) + \mu +$ ☝$\sigma^2)/\sigma$.

Using these expressions, we have that

$$\mu_Y/\mu_Z = \Phi(\delta_1) - (K_{t-1}/Z_{t-1})(\Phi(\delta_2)-\Phi(d_2))(e^{\mu}-e^{r})^{-1} + e^{r}(\Phi(\delta_1)-\Phi(d_1))(e^{\mu}-e^{r})^{-1}$$

and

$$\sigma_Y/\sigma_Z=[e^{2\mu}\exp\{\sigma^2\}\Phi(\delta_0)+(K_{t-1}/Z_{t-1})^2\Phi(-\delta_2)-(e^{\mu}\Phi(\delta_1)+(K_{t-1}/Z_{t-1})\Phi(-\delta_2))^2]^{½}/e^{\mu}(\exp\{\sigma^2\}-1)^{½}$$

5.9 Approximations in a risk neutral environment

The above expressions are clearly difficult to interpret. However, in a risk neutral framework (i.e when $\mu = r$) we can make good approximations to σ_Y/σ_Z. In Appendix A.3 we show that $\sigma_Y/\sigma_Z = ½(\Delta+\Delta^{½})$ and it is easy to show that $\mu_X = \mu_Y = \mu_Z = 0$. Therefore, the option behaves just like a partially hedged portfolio, and in fact they are related by $\alpha = ½(\Delta+\Delta^{½})$.

6. A question of arbitrage

When the environment is not risk neutral (so that $\mu > r$) it appears that while the risk of the option protected portfolio is related to the risk of the portfolio hedged with forwards by the equation $\alpha = ½(\Delta+\Delta^{½})$ with Δ calculated using μ instead of r, the expected returns are not. The expected excess return of the options is less than the forwards. Both portfolios have the same risk but one has a greater return suggesting an arbitrage opportunity. For example, when μ is one and a half standard deviations away from r, where the volatility is 10% and r is zero, the volatility of an at-the-money put protected portfolio is equal to the volatility of a 40% hedged portfolio. Yet the expected return of the put protected portfolio is 4.1% whereas the expected return of the hedged portfolio is 9.7%.

The inefficiency of the option protected portfolio is seen in Chart 4 where a lower expected return than the forward hedged portfolio is evident. The loss of return appears greatest at the most asymmetric part of the curve. It may be that while the long run option protected portfolio return is normally distributed, there is an expected return cost for having the shorter term protection. Further work is needed in this area.

7. Conclusion

We have shown that the rolling of option strategies converges in the long run to a symmetric payoff that can be matched by hedging using forwards. We have seen that the relationship between the hedge ratio, h, and the portfolio delta, Δ, is $1\text{-}h = \frac{1}{2}(\Delta+\Delta^{\frac{1}{2}})$.

In a risk neutral environment this result greatly simplifies the decision on how to include options in a strategic study, as it means we can derive the appropriate hedge ratio and then choose whether to use forwards or options, being aware that they produce identical return distributions.

If the expected return of the asset is considerably greater than the risk free rate then the long run return of the option protected portfolio will be identical in distribution to the portfolio hedged with forwards but with a reduced expected return.

Appendix

A.1 The proof that all three long run portfolio returns are normally distributed

The long run returns of all three portfolios are normally distributed, and the annualised expected return and risk of each portfolio are equal to the single-period expected return and risk.

Proof

The proof is based around the Central Limit Theorem that says that for a sequence of independently identically distributed random variables with finite mean and variance, the sum of these random variables tends to a normal distribution with mean and variance equal to the sum of the individual means and variances. (See Lindgren 1976 for details)

The Central Limit Theorem will hold for all of our three portfolios provided $\ln(Z_t / Z_{t-1})$, $\ln(X_t / X_{t-1})$, $\ln(Y_t / Y_{t-1})$ are each sequences of independent and identically distributed random variables, and provided that their means and variances are finite. We proceed by showing that these criteria are satisfied for all three portfolios.

<u>Portfolio 1:</u>

Portfolio 1 consists solely of the risky asset, and so by definition satisfies the above criteria.

<u>Portfolio 2:</u>

To analyse portfolio 2 we recall from Section 5.2 that the portfolio price process is given as

$$X_t / X_{t-1} = (1 - \alpha) f_{t,t-1} / Z_{t-1} + \alpha\, Z_t / Z_{t-1}.$$

It is useful to note that $f_{t,t-1} = Z_{t-1}\, e^{r}$ (see Hull 1993) where r is the risk free rate, or in the case of currencies r is the differential of the domestic and foreign risk free rates. We have taken r to be constant, and can then write

$$X_t / X_{t-1} = (1 - \alpha)\, e^{r} + \alpha\, Z_t / Z_{t-1}.$$

First, we show that $\ln(X_t / X_{t-1})$ are independent and identically distributed. Given that Z_t / Z_{t-1} are independently and identically distributed, and that α and r are constants, we immediately have that X_t / X_{t-1} are independent and identically distributed. It follows immediately that the same must then also be true for $\ln(X_t / X_{t-1})$.

Now we show that $E[\ln(X_t / X_{t-1})]$ and $V[\ln(X_t / X_{t-1})]$ are both finite. Given that $\ln(x) < x-1$ (see Abramowitz and Stegun 1972) we have upper bounds for $E[\ln(X_t / X_{t-1})]$ and $E[\ln(X_t / X_{t-1})^2]$ which are $E[\ln(X_t / X_{t-1})] < E[X_t / X_{t-1} - 1]$ and $E[\ln(X_t / X_{t-1})^2] < E[(X_t / X_{t-1} - 1)^2]$. Therefore, to conclude that the mean and variance of $\ln(X_t / X_{t-1})$ must both be finite, we are required to prove that $E[X_t / X_{t-1} - 1]$ and $E[(X_t / X_{t-1} - 1)^2]$ are finite. We know from Section 5.4 that $E[Z_t / Z_{t-1}]$ and $V[Z_t / Z_{t-1}]$ are finite, and since X_t / X_{t-1} is a linear combination of Z_t / Z_{t-1} and a constant, $E[X_t / X_{t-1}]$ and $V[X_t / X_{t-1}]$ are also finite, and so it follows that $E[X_t / X_{t-1} - 1]$ and $E[(X_t / X_{t-1} - 1)^2]$ must also be finite.

<u>Portfolio 3:</u>

Recalling once again from Section 5.2, the third portfolio price process is given by

$$Y_t / Y_{t-1} = \max(Z_t / Z_{t-1}, K_{t-1} / Z_{t-1}) + p_{t-1}\, e^{r} / Z_{t-1}.$$

We start by showing that $\ln(Y_t / Y_{t-1})$ are independent and identically distributed. To do this we must show that K_{t-1} / Z_{t-1} and $p_{t-1}\, e^{r} / Z_{t-1}$ are constant through time. The variable K_{t-1} is derived from the initial delta of the portfolio, which is constant for each option. The delta of the portfolio, denoted by Δ, is equal to one plus the delta of the put, and thus

$$\Delta = \Phi[\ (\ln(Z_{t-1}/K_{t-1}) + r + \tfrac{1}{2}\sigma^2)/\sigma]$$

where Φ[ ] denotes the standardised normal cumulative probability distribution function. Given the Δ, r and σ are all constant, Z_{t-1}/K_{t-1} is also constant.

Additionally, through the Black Sholes pricing formula (see Hull 1993), we have that

$$p_{t-1}e^r / Z_{t-1} = K_{t-1}/Z_{t-1}\ \Phi[\ -(\ln(Z_{t-1}/K_{t-1}) + r - \tfrac{1}{2}\sigma^2)/\sigma] - e^r\ \Phi[\ -(\ln(Z_{t-1}/K_{t-1}) + r + \tfrac{1}{2}\sigma^2)/\sigma].$$

Now, we have already shown that Z_{t-1}/K_{t-1} is constant, and so we can immediately see that $p_{t-1}e^r / Z_{t-1}$ is also a constant. These are important characteristics because it means that the Y_t / Y_{t-1} are identically distributed, and since they only involve the corresponding contemporaneous value of Z_t / Z_{t-1}, they are also mutually independent. It follows immediately that $\ln(Z_t / Z_{t-1})$ are independent and identically distributed.

We continue by showing that $E[\ln(Y_t / Y_{t-1})]$and $V[\ln(Y_t / Y_{t-1})]$ are finite. As in the second portfolio example, we proceed by showing that $E[Y_t / Y_{t-1} - 1]$ and $V[Y_t / Y_{t-1} - 1]$ are finite then use the fact that $\ln(x) < x-1$ to immediately give us that $E[\ln(Y_t / Y_{t-1})]$and $V[\ln(Y_t / Y_{t-1})]$ are finite. Clearly constants have finite mean and variance therefore if $\max(Z_t / Z_{t-1}, K_{t-1} / Z_{t-1})$ has a finite mean and variance then so does Y_t / Y_{t-1}. The variable Z_t always positive for all t, therefore $\max(Z_t / Z_{t-1}, K_{t-1} / Z_{t-1}) < Z_t / Z_{t-1} + K_{t-1} / Z_{t-1}$, and so $E[\max(Z_t / Z_{t-1}, K_{t-1} / Z_{t-1})] < E[Z_t / Z_{t-1} + K_{t-1} / Z_{t-1}]$and $E[\max(Z_t / Z_{t-1}, K_{t-1} / Z_{t-1})^2] < E[(Z_t / Z_{t-1} + K_{t-1} / Z_{t-1})^2]$. Following the upper bound login used previously, all that remains for us to do is to show that these two upper bounds are finite. We can see that provided that $0<\Delta<1$, then K_{t-1} must assume a finite value. We know that E[Z_t / Z_{t-1}] is finite from Section 5.4 and since we have just shown that K_{t-1} is finite we must have that E[Z_t / Z_{t-1} + K_{t-1} / Z_{t-1}] is finite. An identical argument also holds for E[(Z_t / Z_{t-1} + K_{t-1} / $Z_{t-1})^2$].

A.2 Deriving the mean and variance of the option protected portfolio

In order to clarify the algebra, all variables valued at time $t-1$ appear without the time index (i.e. Z_{t-1} appears as Z).

A.2(i) Deriving the mean

To derive an expression for $E[Y_t/Y - 1]$ we must first derive an expression for $E[\max(Z_t,K)]/Z$ and approach this by stating that

$$E[\max(Z_t,K)]/Z = I_1 + I_2$$

where

$$I_1 = \int_{K/Z}^{\infty} \frac{x}{\sqrt{2\pi}\sigma x} \exp\{-½(\ln x - \theta)^2/\sigma^2\}dx \text{ and } I_2 = \int_{-\infty}^{K/Z} \frac{K/Z}{\sqrt{2\pi}\sigma x} \exp\{-½(\ln x - \theta)^2/\sigma^2\}dx.$$

where $\theta = \mu + ½\sigma^2$.

Addressing I_1, by letting $z = (\ln x - \theta)/\sigma$ we have that

$$I_1 = \int_{(\ln(K/Z)-\theta)/\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\{-½(z-\sigma)^2\}dz \exp\{\theta + ½\sigma^2\}$$

$$= \exp\{\theta + ½\sigma^2\} \int_{(\ln(K/Z)-\theta-\sigma^2)/\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\{-½z^2\}dz$$

$$= e^{\mu}(1 - \Phi(-\delta_1)) \qquad \text{where } \delta_1 = (\ln(Z/K) + \mu + ½\sigma^2)/\sigma$$

$$= e^{\mu}\Phi(\delta_1)\,.$$

Addressing I_2, by letting $z = (\ln x - \theta)/\sigma$ we have that

$$I_2 = \int_{-\infty}^{(\ln(K/Z)-\theta)/\sigma} \frac{K/Z}{\sqrt{2\pi}} \exp\{-½z^2\}dz$$

$$= (K/Z)\Phi(-\delta_2) \qquad \text{where } \delta_2 = (\ln(Z/K) + \mu - ½\sigma^2)/\sigma = \delta_1 - \sigma\,.$$

Substituting the solutions for I_1 and I_2 into the expectation expression we have that

$$E[\max(Z_t K)]/Z = e^{\mu}\Phi(\delta_1) + (K/Z)\Phi(-\delta_2).$$

Directly from option pricing theory, following similar algebra to the above, we have that

$$e^r p/Z = e^r\Phi(d_1) + (K/Z)\Phi(-d_2) - e^r$$

where $d_1 = (\ln(Z/K) + r + ½\sigma^2)/\sigma$ and $d_1 = (\ln(Z/K) + r - ½\sigma^2)/\sigma$. Putting all this together we have that

$$E[Y_t/Y - 1] = e^{\mu}\Phi(\delta_1) + (K/Z)\Phi(-\delta_2) - (e^r\Phi(d_1) + (K/Z)\Phi(-d_2) - e^r) - 1.$$

A.2(ii) Deriving the variance

To derive an expression for $V[Y_t/Y\text{-}1]$ we note that $V[Y_t/Y - 1] = V[\max(Z_t,K)/Z]$, and so we must first derive an expression for $E[\max(Z_t^2,K^2)]/Z^2$. We approach this by stating that

$$E[\max(Z_t^2,K^2)]/Z^2 = I_3 + I_4$$

where

$$I_3 = \int_{K/Z}^{\infty} \frac{x^2}{\sqrt{2\pi}\sigma x} \exp\{-½(\ln x - \theta)^2/\sigma^2\}dx \text{ and } I_4 = \int_{-\infty}^{K/Z} \frac{(K/Z)^2}{\sqrt{2\pi}\sigma x} \exp\{-½(\ln x - \theta)^2/\sigma^2\}dx\,.$$

Addressing I_3, by letting $z = (\ln x - \theta)/\sigma$ we have that

$$I_3 = \int_{(\ln(K/Z)-\theta)/\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\{-½(z - 2\sigma)^2\}dz \exp\{2\theta + 2\sigma^2\}$$

$$= \exp\{2\theta + 2\sigma^2\} \int_{(\ln(K/Z)-\theta-2\sigma^2)/\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\{-½z^2\}dz$$

$$= (e^{\mu})^2 e^{\sigma^2}(1-\Phi(-\delta_0)) \qquad \text{where } \delta_0 = (\ln(Z/K) + \mu + e\sigma^2)/\sigma = \delta_1 + \sigma$$

$$= (e^{\mu})^2 e^{\sigma^2}\Phi(\delta_0).$$

Addressing I_4, by letting $z = (\ln x - \theta)/\sigma$ we have that

$$I_4 = \int_{-\infty}^{(\ln(K/Z)-\theta)/\sigma} \frac{(K/Z)^2}{\sqrt{2\pi}} \exp\{-½z^2\}dz$$

$$= (K/Z)^2\Phi(-\delta_2).$$

Substituting the solutions for I_3 and I_4 into the expectation expression we have that

$$E[\max(Z_t^2, K^2)]/Z^2 = (e^{\mu})^2 \exp\{\sigma^2\}\Phi(\delta_0) + (K/Z)^2\Phi(-\delta_2).$$

Combining this with our expression for $E[\max(Z_t,K)]/Z$, and using the fact that $V[\max(Z_t,K)] = E[\max(Z_t^2,K^2)]/Z^2 - (E[\max(Z_t,K)]/Z)^2$ we have that

$V[Y_t/Y - 1] = (e^{\mu})^2\exp\{\sigma^2\}\Phi(\delta_0) + (K/Z)^2\Phi(-\delta_2) - (e^{\mu}\Phi(\delta_1) + (K/Z)\Phi(-\delta_2))^2.$

A.3 Approximating the volatility of the option protected portfolio

The volatility of the option protected portfolio expressed as a proportion of the volatility of the underlying asset portfolio is approximately $½(\Delta + \Delta^{½})$.

Proof

The ratio of volatilities can be expressed as a function of Δ, the delta of the protected portfolio. To prove that $½(\Delta + \Delta^{½})$ is an approximation of the ratio we show that it is the second order Taylor expansion (see Abramowitz and Stegun 1970) in terms of $\Delta^{½}$ taken around $\Delta=½$. Letting $f = \sigma_Y/\sigma_Z$ we note that this expansion gives us that

$$f(\Delta^{1/2}) = f(\Delta_0^{1/2}) + f'(\Delta_0^{1/2})(\Delta^{1/2}-\Delta_0^{1/2}) + \tfrac{1}{2}f''(\Delta_0^{1/2})(\Delta^{1/2}-\Delta_0^{1/2})^2$$

$$= [\, f(\Delta_0^{1/2}) - f'(\Delta_0^{1/2})\Delta_0^{1/2} + \tfrac{1}{2}f''(\Delta_0^{1/2})\Delta_0 \,] + [\, f'(\Delta_0^{1/2}) - f''(\Delta_0^{1/2})\Delta_0^{1/2} \,]\,\Delta^{1/2} + \tfrac{1}{2}f''(\Delta_0^{1/2})\Delta.$$

We shall take $\Delta_0 = ½$, and thus can express

$$f(\Delta^{1/2}) = [\, f(½^{1/2}) - ½^{1/2}f'(½^{1/2}) + ¼f''(½^{1/2}) \,] + [\, f'(½^{1/2}) - ½^{1/2}f''(½^{1/2}) \,]\,\Delta^{1/2} + ½f''(½^{1/2})\Delta.$$

All that remains is to derive $f(½^{1/2})$, $f'(½^{1/2})$, and $f''(½^{1/2})$.

Before proceeding we make some notational comments. Recall that $f = \sigma_Y/\sigma_Z$ and

$$\sigma_Y^2 = V[Y_t/Y_{t-1}-1] = V[\max(K_{t-1},Z_t)/Z_{t-1} - e^r p_{t-1}/Z_{t-1}] = V[\max(K_{t-1}/Z_{t-1}, Z_t/Z_{t-1})].$$

So we can write that

$$\sigma_Y^2 = E_2 - E_1^2$$

where $E_1 = E[\max(K_{t-1}/Z_{t-1}, Z_t/Z_{t-1})]$ and $E_2 = E[\max(K_{t-1}/Z_{t-1}, Z_t/Z_{t-1})^2]$.

We shall also express δ_0 and δ_2 as functions of δ_1, so that our equations will only involve δ_1, which we shall abbreviate to δ.

<u>Deriving $f(½^{1/2})$:</u>

First we note from Appendix A.2 that

$$E_1 = \exp\{\mu\}[\Phi(\delta) + \exp\{½\sigma^2-\sigma\delta\}\ (1-\Phi(\delta-\sigma))]$$

and

$$E_2 = \exp\{2\mu+\sigma^2\}\ [\Phi(\delta+\sigma) + \exp\{-2\sigma\delta\}\ (1-\Phi(\delta-\sigma))].$$

When $\Delta = ½$, and thus when $\delta = 0$ we have that

$$E_1 = \exp\{\mu\}[½ + \exp\{½\sigma^2\}\Phi(\sigma)] \text{ and } E_2 = \exp\{2\mu+\sigma^2\}2\Phi(\sigma),$$

additionally

$$E_1^2 = \exp\{2\mu\}[\tfrac{1}{4} + \exp\{\sigma^2\}\Phi^2(\sigma) + \exp\{\tfrac{1}{2}\sigma^2\}\Phi(\sigma)].$$

We can combine these terms to give us that

$$\sigma_Y^2 = \exp\{2\mu\}Q$$

where $Q = \exp\{\sigma^2\}\Phi(\sigma)(2-\Phi(\sigma)) - \exp\{\tfrac{1}{2}\sigma^2\}\Phi(\sigma) - \tfrac{1}{4}$.

We can rewrite this as

$$\sigma_Y^2 = \sigma_Z^2\, Q/(\exp\{\sigma^2\}-1)$$

and so

$$f(\tfrac{1}{2}^{\frac{1}{2}}) = [Q/(\exp\{\sigma^2\}-1)]^{\frac{1}{2}}.$$

We can see that this $f(\tfrac{1}{2}^{\frac{1}{2}})$ is a function solely of σ, and now examine it over a typical range of σ. When $\sigma=0$, $f(\tfrac{1}{2}^{\frac{1}{2}})=0.58$ and when $\sigma=30\%$, $f(\tfrac{1}{2}^{\frac{1}{2}})=0.63$. It appears monotonic over this range. In conclusion we note that this expression is almost constant and make the approximation that $f(\tfrac{1}{2}^{\frac{1}{2}}) \approx 0.6$, which we shall express algebraically as $[\tfrac{1}{4}(1+\text{🕐})]^2$.

Substituting this into our original expression, we have that

$$f(\tfrac{1}{2}^{\frac{1}{2}}) \approx \tfrac{1}{4}(1+\text{🕐}).$$

<u>Deriving $f'(\tfrac{1}{2}^{\frac{1}{2}})$:</u>

To begin we have that

$$f'(\Delta^{\frac{1}{2}}) = (d\sigma_Y/d\Delta^{\frac{1}{2}})/\sigma_Z$$

which can be written in chain rule form as

$$d\sigma_Y/d\Delta^{\frac{1}{2}} = d\sigma_Y/d\sigma_Y^2 \times d\sigma_Y^2/d\delta \times d\delta/d\Delta \times d\Delta/d\Delta^{\frac{1}{2}}.$$

Trivially we have that $d\sigma_Y/d\sigma_Y^2 = 2\sigma_Y$, $d\Delta/d\Delta^{\frac{1}{2}} = 2\Delta^{\frac{1}{2}}$, and also that $\Delta = \Phi(\delta)$ so that $d\Delta/d\delta = (2\pi)^{-1}\exp\{-\tfrac{1}{2}\delta^2\}$. Using our previous notation, we have that $\sigma_Y^2 = E_2 - E_1^2$ so that $d\,\sigma_Y^2/d\delta = dE_2/d\delta - 2E_1\, dE_1/d\delta$

All this gives us that

$$d\sigma_Y/d\Delta^{1/2} = \Delta^{1/2}/\sigma_Y\ (2\pi)^{1/2}\ \exp\{½\delta^2\}(dE_2/d\delta - 2E_1 dE_1/d\delta).$$

The last two derivatives are written as

$$dE_1/d\delta = -\exp\{\mu+½\sigma^2-\sigma\delta\}\sigma\Phi(\sigma-\delta)$$

and

$$dE_2/d\delta = -2\exp\{2\mu+\sigma^2-2\sigma\delta\}\sigma\ \Phi(\sigma-\delta).$$

When $\Delta=½$, and thus when $\delta=0$ we have that

$$dE_1/d\delta = -\exp\{\mu+½\sigma^2\}\sigma\Phi(\sigma) \text{ and } dE_2/d\delta = -2\exp\{2\mu+\sigma^2\}\sigma\Phi(\sigma)$$

which gives us that

$$d\sigma_Y^2/d\delta = 2\exp\{2\mu+\sigma^2\}\sigma\Phi(\sigma)[\ ½\exp\{-½\sigma^2\}+\Phi(\sigma)-1]$$

and so

$$d\sigma_Y/d\Delta^{1/2} = 2\pi^{1/2}\ \exp\{2\mu+\sigma^2\}\sigma\Phi(\sigma)[\ ½\exp\{-½\sigma^2\}+\Phi(\sigma)-1]/\ \sigma_Y.$$

Now, by noting that $\sigma_Z^2 = e^{2\mu}(\exp\{\sigma^2\}-1)$, and from our previous calculation that $\sigma_Y\sigma_Z=\sigma_Z^2\ [Q / (\exp\{\sigma^2\}-1)]^{1/2}$, we can write

$$f'(½^{1/2}) = 2\pi^{1/2}\ \exp\{\sigma^2\}\sigma\Phi(\sigma)[\ ½\exp\{-½\sigma^2\}+\Phi(\sigma)-1] / [Q\ (\exp\{\sigma^2\}-1)]^{1/2}.$$

As before we note that $f'(½^{1/2})$ is a function only of σ and observe it over a typical range of σ. When $\sigma=0$, $f'(½^{1/2})=1.21$ and when $\sigma=30\%$, $f'(½^{1/2})=1.05$. It appears monotonic over this range. Again we note that this expression is very close to being constant, and make the approximation that $f'(½^{1/2})\approx1.207$ which we shall express algebraically as $½+½^{1/2}$.

Deriving $f''(½^{1/2})$:

We shall let H denote $d\sigma_Y/d\Delta^{1/2}$ and, from our previous section, we have that

$$H = d\sigma_Y^2/d\delta\ \sigma_Y^{-1}\ (2\pi)^{1/2}\exp\{½\delta^2\}\Phi(\delta)^{1/2}.$$

Using the chain rule again, we can write

$$dH/d\Delta^{½} = dH/d\delta \times d\delta/d\Delta \times d\Delta/d\Delta^{½}$$

$$= dH/d\delta\ (2\pi)^{½}\exp\{½\delta^2\}2\Phi(\delta)^{½}.$$

We must address $dH/d\delta$, and have that

$$dH/d\delta = (2\pi)^{½}\exp\{½\delta^2\}\Phi(\delta)^{½}/\sigma_Y[d^2\sigma_Y^2/d\delta^2 - ½(d\sigma_Y^2/d\delta)^2/\sigma_Y^2]$$

$$+ d\sigma_Y^2/d\delta\ [(2\pi)^{½}\exp\{½\delta^2\}\Phi(\delta)^{½}\delta + ½\Phi(\delta)^{-½}]/\sigma_Y,$$

which gives us that

$$dH/d\Delta^{½} = 4\pi\ \exp\{\delta^2\}\Phi(\delta)/\sigma_Y[d^2\sigma_Y^2/d\delta^2 - ½(d\sigma_Y^2/d\delta)^2/\sigma_Y^2]$$

$$+ (2\pi)^{½}\exp\{½\delta^2\}d\sigma_Y^2/d\delta\ [2(2\pi)^{½}\exp\{½\delta^2\}\Phi(\delta)\delta+1]/\sigma_Y.$$

When $\Delta=½$, and thus when $\delta=0$ we have that

$$dH/d\Delta^{½} = 2\pi/\sigma_Y[d^2\sigma_Y^2/d\delta^2 - ½(d\sigma_Y^2/d\delta)^2/\sigma_Y^2] + (2\pi)^{½}(d\sigma_Y^2/d\delta)/\sigma_Y$$

which, by noting that $\sigma_Y\sigma_Z=\sigma_Z^2\ [Q\,/\,(\exp\{\sigma^2\}-1)]^{½}$, gives us that

$$f'(½) = [2\pi(d^2\sigma_Y^2/d\delta^2 - ½(d\sigma_Y^2/d\delta)^2/\sigma_Y^2) + (2\pi)^{½}d\sigma_Y^2/d\delta]/\,(\sigma_Z[Q\ (\exp\{\sigma^2\}-1)]^{½}).$$

All that remains is to expand the derivatives of σ_Y^2. We begin this by recalling that

$$\sigma_Y^2 = E_2-E_1^2$$

where $E_1=e^{\mu}[\Phi(\delta)+\exp\{½\sigma^2\}e^{-\sigma\delta}\Phi(\sigma-\delta)]$ and $E_2=e^{2\mu}\exp\{\sigma^2\}[\Phi(\sigma+\delta)+e^{-2\sigma\delta}\Phi(\sigma-\delta)]$.

The first derivative is then given by

$$d\sigma_Y^2/d\delta = dE_2/d\delta - 2E_1dE_1/d\delta$$

where $dE_1/d\delta = -e^{\mu}\exp\{½\sigma^2\}e^{-\sigma\delta}\sigma\Phi(\sigma-\delta)$ and $dE_2/d\delta = -2e^{2\mu}\exp\{\sigma^2\}e^{-2\sigma\delta}\sigma\Phi(\sigma-\delta)$.

The second derivative is then given by

$$d^2\sigma_Y^2/d\delta^2 = d^2E_2/d\delta^2 - 2(dE_1/d\delta)^2 - 2E_1\,d^2E_1/d\delta^2$$

where

$$d^2E_1/d\delta^2 = e^{\mu}\exp\{½\sigma^2\}e^{-\sigma\delta}\sigma[\Phi(\sigma-\delta)\sigma + (2\pi)^{-½}\exp\{-½(\sigma-\delta)^2\}] \text{ and}$$

$$d^2E_2/d\delta^2 = 2e^{2\mu}\exp\{\sigma^2\}e^{-2\sigma\delta}\sigma[2\Phi(\sigma-\delta)\sigma + (2\pi)^{-½}\exp\{-½(\sigma-\delta)^2\}].$$

When $\Delta=½$, and thus when $\delta=0$ we can combine the above expressions to show that

$d\sigma_Y^2/d\delta = 2\, e^{2\mu}\exp\{\sigma^2\}\sigma\Phi(\sigma)[½\exp\{-½\sigma^2\}+\Phi(\sigma)-1]$ and

$$d^2\sigma_Y^2/d\delta^2 = 2e^{2\mu}\exp\{\sigma^2\}\sigma[2\Phi(\sigma)\sigma + (2\pi)^{-½}\exp\{-½\sigma^2\} - \sigma\Phi(\sigma)^2 - (½\exp\{-½\sigma^2\}+\Phi(\sigma)) \times (\Phi(\sigma)\sigma + (2\pi)^{-½}\exp\{-½\sigma^2\})]$$

We have completed the derivation of $f'(½^½)$, and by substituting $e^{2\mu}=\sigma_Z^2/(\exp\{\sigma^2\}-1)$ we have that

$f'(½^½) = \exp\{\sigma^2\}\sigma\ (4\pi\ [2\Phi(\sigma)\sigma + (2\pi)^{-½}\exp\{-½\sigma^2\} - \sigma\Phi(\sigma)^2 - (½\exp\{-½\sigma^2\}+\Phi(\sigma)) \times (\Phi(\sigma)\sigma + (2\pi)^{-½}\exp\{-½\sigma^2\}) - \Phi(\sigma)\sigma\ \exp\{\sigma^2\}(½\exp\{-½\sigma^2\}+\Phi(\sigma)-1)^2/Q] + (2\pi)^{½}\ \Phi(\sigma)\sigma\ 2(½\exp\{-½\sigma^2\}+\Phi(\sigma)-1))/\ [Q\ (\exp\{\sigma^2\}-1)]^{½}$.

This is a very messy expression, although we do note that it is a function only of σ. We therefore observe it over a typical range of σ. When $\sigma=0$, $f'(½^½)=1.16$ and when $\sigma=30\%$, $f'(½^½)=0.69$. It appears monotonic over this range. There is slightly more variation in f' as a function of σ, but is still fairly close to 1. We make the approximation that $f'(½^½)\approx 1$.

<u>Constructing the Taylor series:</u>

From above, the first term in the Taylor expansion is given as

$$f(½^½) - ½^½ f'(½^½) + ¼f'(\ ½^½) = ¼(1+\oplus) - ½^½(½+½^½) + ¼ = 0.$$

The second term in the Taylor expansion is given as

$$f'(½^½) - ½^½ f'(½^½) = ½+½^½ - ½^½ = ½.$$

The third term in the Taylor expansion is given as

$$\tfrac{1}{2}f'(\tfrac{1}{2}^{\frac{1}{2}}) = \tfrac{1}{2}.$$

Therefore the full approximation is given as

$$\tfrac{1}{2}\,(\Delta + \Delta^{\frac{1}{2}}).$$

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Chart 1. Caption

There is a one-to-one mapping between the long run return distribution of a portfolio hedged with forwards, and an option protected portfolio. The relationship is approximately $\alpha = \frac{1}{2}(\Delta+\Delta^{1/2})$ where α is the amount of exposure after hedging and Δ is the initial delta of the option protected portfolio. The solid line is the actual relationship, the dotted line is the approximation.

Chart 2. Caption

The has been no significant difference between hedging with forwards or options when viewing empirical results from 1988 to 2000.

Chart 3. Caption

Simulation illustrates how a sequence of option returns converges in time to a symmetric distribution equivalent to a fixed hedge portfolio. After 10 years there is no distinguishable difference in return distribution from an at-the-money put protected portfolio and a 60% hedged portfolio.

Chart 4. Caption

When the mean of the risky asset is considerably different from the risk free rate, then the option based portfolio underperforms the forward based portfolio.